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Article · January 2016

DOI: 10.9734/BJMCS/2016/25684

K. M. Khalil
Faculty of science, Benha university

Amira Masoud Omran
Benha University

British Journal of Mathematics & Computer Science
16(4): 1-12, 2016, Article no.BJMCS.25684
ISSN: 2231-0851

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Periodic Solutions of a Modified Duffing Equation Subjected to a Bi-Harmonic Parametric and External Excitations

A. M. Elnaggar[1], A. F. El-Bassiouny[1], K. M. Khalil[1] and A. M. Omran[1*]

[1] *Department of Mathematics, Faculty of Science, Benha University, B.O. 13518, Egypt.*

Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/BJMCS/2016/25684
Editor(s):
(1) Andrej V. Plotnikov, Department of Applied and Calculus Mathematics and CAD, Odessa State Academy of Civil Engineering and Architecture, Ukraine.
Reviewers:
(1) Anonymous, University of Hacettepe, Ankara, Turkey.
(2) Andrej Kon'kov, Moscow Lomonosov State University, Russia.
Complete Peer review History: http://sciencedomain.org/review-history/14529

Received: 15th March 2016
Accepted: 26th April 2016
Published: 9th May 2016

Original Research Article

Abstract

In this paper, we investigated the periodic solutions of type superharmonic and subsuperharmonic of modified Duffing equation subjected to a bi-harmonic parametric and external excitations. The method of multiple scales is used to construct a first order uniform expansion of approximate solutions. Two first-order nonlinear ordinary differential equations(Modulation Equation) are derived from the evolution of the amplitude and the phase. Steady state solutions and their stability are given for selected values of the system parameters. The consequences of these (quadratic and cubic) nonlinearities on these the vibrations are particularly examined. With this research, it has been confirmed that the qualitative effects of these nonlinearities are different. Regions of the hard (soft) nonlinearity of the system exist for the case of subsuperharmonic oscillation. Numerical solutions are presented in a group of figures which demonstrate the actions of the steady-state reaction plenitude as the purpose of the detuning parameter.

**Corresponding author: E-mail: amiramasoud.am@gmail.com;*

Keywords: MEMS; weakly nonlinear differential equation; multiple scales method; parametric excitation and external excitation.

2016 Mathematics Subject Classification: 34K13-34K27.

1 Introduction

In the past few years, many more of the statistical methods were used to resolve a wide variety of statistical, physical and technological innovation problems straight line and nonlinear.

In the present study, we use the method of multiple scales(MMS) for determination of the response of nonlinearly oscillator to external excitation. For an extensive review, we relate your reader to [1, 2, 3, 4].

Zavodney et al. [5] studied the response of a model includes quadratic and cubic geometric nonlinearities. They found that stable limit cycles could exist. Zavodney and Nayfeh [6] investigated the dynamics of a cantilever beam carrying a lumped mass. They modeled the structure with cubic geometric and inertia nonlinearities. A thorough analysis of the governing equation of the motion has provided an accurate model of the dynamic response of such devices [7, 8, 9], which has been compared well with experimental results. The method of multiple scales is applied throughout. Asfar[10] took material nonlinearity into consideration in the analysis of the performance of an elastomeric damper with a spring Harding cubic effects near primary resonance condition applying multiple scale method. Kamel and Amer [11] studied the behavior of one-degree-of-freedom system with different quadratic damping and cubic stiffness nonlinearities simulating the axial vibration of a cantilever beam under multi-parametric excitation forces.The method of multiple scales has been used to solve the equations to first order perturbation. Eissa and Amer [12] studied the vibrations of a second purchase program to the first method of a cantilever ray exposed to both exterior and parametric excitation at main and subharmonic solutions. Nayfeh [13] compared use of the way of several machines with reconstitution and the general way of calculating for identifying higher-order estimates of three single-degree-of-freedom systems and a two-degree-of-freedom system. He showed that the second-order frequency-response equation possesses spurious solutions for the case of softening nonlinearity. El-Bassiouny [14] investigated the effects of quadratic and cubic nonlinearities in elastomeric content dampers on torsional vibrations management. The multiple time scales is used to solve the stability equations at primary resonance. The multiple-scale perturbation technique is applied throughout. A limit value of straight line damping has been acquired, where the program vibrations can be decreased considerably. Masana and Daqaq [15] have carried out detailed studies of the post-buckled piezoelectric beam. However, the advantage of the bistable device over the linear device was not uniform, with the exception at very low frequencies when the bistable harvester was excited into high-energy orbits but the linear harvester was weakly excited. Superharmonic dynamics were specifically considered in a series of comparable tests and simulations [16]. Sebald et al. [17]described a similar technique whereby an impulsive voltage could be applied to the harvesting circuit to achieve the same objective theoretically. This reduces the computational cost since the electrostatic force term in the discretized equation will not require complicated numerical integration (integrating a numerator term over a denominator term numerically is computationally expensive) [18].

The issue of parametric resonance occurs in many divisions of science and technological innovation. One of the essential issues is that of powerful uncertainty. There are cases in which the influence of a small vibration loading can stabilize a system which is statically unstable and vice-versa. There are many books devoted to the analysis and applications of the problem of parametric excitation [19]. As an example McLachlan [20] discussed the theory and applications of the Mathieu functions. The interfacial stability with periodic forces is a relatively new topic in the theory of hydrodynamic stability. The statistical research is more challenging because: (a) the method of normal modes is not applicable and (b) the linearized differential equations have time-dependent coefficients so that, the exponential time dependence of the perturbation is not separable. Elhefnawy and El-Bassiouny [21] studied the nonlinear stability and chaos in Electrohydrodynamics. El-Bassiouny [22] investigated the principal parametric resonance of a single-degree-of-freedom system with nonlinear two-frequency parametric and self-excitations. Qualitative research and asymptotic development techniques are employed to estimate the use of steady-state reactions. The impact of damping, magnitudes of nonlinear excitation and self-excitation are examined. El-Bassiouny and Eissa [23] aanalyzed the behavior of two-degrees-of-freedom vibrating mechanical structure, which is described by two nonlinear differential equations with quadratic and cubic nonlinearities, subjected to multi-frequency parametric excitations in the presence of two-to-one internal resonance. Two estimated methods (the multiple scales and the generalized synchronization) are used to obtain a uniform first-order expansion. The results achieved by the two methods are in excellent agreement. Elnaggar et al. [24] studied harmonic and subharmonic resonance of micro-electro-mechanical system (MEMS) subjected to a weakly nonlinear parametric and external excitation. Elnaggar et al. [25] used the method of multiple scales to investigated the saddle-node bifurcation control for an odd nonlinearity problem. Elnaggar et al. [26] analyzed the perturbation analysis of an electrostatic micro-electro-mechanical system(MEMS) subjected to external and nonlinear parametric excitations. Harmonic, subharmonic and superharmonic resonance of a weakly nonlinear dynamical program exposed to exterior excitation and parametric excitation or both are examined by Elnaggar et al.[27] and [28].

In this paper, an analysis of superharmonic oscillation of order two and subsuperharmonic oscillation of order three-to-two are illustrated. Two first-order nonlinear ordinary differential equations are derived for the evolution of the amplitude and phase with damping, nonlinearity, and all possible solutions based on mathematically justified multiple scales method. Stability analysis is carried out for each case.

2 Perturbation Analysis

The mathematical model of the micro-electro-mechanical systems (MEMS) is represented by the following weakly nonlinear second order differential equation

$$\begin{aligned} &u'' + 2\epsilon\mu u' + \omega_o^2 u + \epsilon(\alpha_1 u^2 + \alpha_2 u^3) - \epsilon\alpha(2u + 3u^2 + 4u^3) - \epsilon(2u + 3u^2 + 4u^3) \\ &(F_1 \cos[\Omega t] + F_2 \cos[2\Omega t]) - \epsilon(\alpha + F_1 \cos[\Omega t] + F_2 \cos[2\Omega t]) = 0. \end{aligned} \tag{2.1}$$

Equation (2.1) represent Duffing formula exposed to weakly nonlinear parametric excitation, where the dots indicate differentiation with respect to t, μ is the coefficient of viscous damping,, ϵ is a small parameter $\epsilon \ll 1$, ω_o is the linear natural frequency, Ω is frequency of the external excitation, α is the coefficient of linear term. α_1 and α_2 are the coefficients of the nonlinear terms. F_1 and F_2 are the coefficients of linear and nonlinear parametric excitations. To determine a first-order uniform expansion of the solutions of Eq.(2.1). Let

$$u(t;\epsilon) = u_o(T_o, T_1) + \epsilon u_1(T_o, T_1) + O(\epsilon^2), \quad T_n = \epsilon^n t, \tag{2.2}$$

where $T_o = t$ is the first scale associated with changes occurring at the frequencies ω_o and Ω, and $T_1 = \epsilon t$ is a slow scale associated with modulations in the amplitude. Denote $D_o = \frac{\partial}{\partial T_o}$ and $D_1 = \frac{\partial}{\partial T_1}$. Substituting Eqs.(2.2) into Eq.(2.1) and equating the coefficients of like power of ϵ, one has the following equations to order O(1) and to order O(ϵ) :

$$D_0^2 u_o + \omega_o^2 u_o = 0. \tag{2.3}$$

$$\begin{aligned} D_o^2u_1+\omega_o^2u_1 =& -2\mu D_ou_o-2D_oD_1u+F_1\cos[\Omega T_o]+F_2\cos[2\Omega T_o]-\alpha_2u_o^3\\ &+2u_o(F_1\cos[\Omega T_o]+F_2\cos[2\Omega T_o])+3\alpha u_o^2+2\alpha u_o+\alpha\\ &+3u_o^2(F_1\cos[\Omega T_o]+F_2\cos[2\Omega T_o])-\alpha_1u_o^2+4\alpha u_o^3\\ &+4u_o^3(F_1\cos[\Omega T_o]+F_2\cos[2\Omega T_o]). \end{aligned} \tag{2.4}$$

The solution of Eq.(2.3) can be expression the form

$$u_o(T_o,T_1) \;=\; A(T_1)e^{i\omega_oT_o}+c.c, \tag{2.5}$$

where A is the amplitude of the response and is a function of T_1 and $c.c$ is the complex conjugate of A, substitute Eq.(2.5)into Eq.(2.4), we get

$$\begin{aligned} D_o^2u_1+\omega_o^2u_1 =& -(-2\alpha A+2i\mu\omega_oA-12\alpha A^2\bar{A}+3A^2\alpha_2\bar{A}+2i\omega_oA')e^{i\omega_oT_o}\\ &+6\alpha A\bar{A}-2A\alpha_1\bar{A}+\frac{3}{2}F_1\bar{A}^2e^{i(\Omega-2\omega_o)T_o}+\frac{3}{2}F_2\bar{A}^2e^{i(2\Omega-2\omega_o)T_o}\\ &+(F_1\bar{A}+6AF_1\bar{A}^2)e^{i(\Omega-\omega_o)T_o}+(F_2\bar{A}+6AF_2\bar{A}^2)e^{i(2\Omega-\omega_o)T_o}\\ &+(\frac{F_1}{2}+3AF_1\bar{A})e^{i\Omega T_o}+(\frac{F_2}{2}+3AF_2\bar{A})e^{2i\Omega T_o}\\ &+2F_1\bar{A}^3e^{i(\Omega-3\omega_o)T_o}+2F_2\bar{A}^3e^{i(2\Omega-3\omega_o)T_o}+NST+c.c. \end{aligned} \tag{2.6}$$

Where the prime stands for the derivative with respect to T_1, overbar represents the complex conjugate and NST stands for nonsecular terms. Any particular solution of equation(2.6) contains secular terms, and it may contain small-divisor terms depending on the solution conditions, it can be seen that solutions occur when $2\Omega \cong \omega_o$ and $3\Omega \cong 2\omega_o$. In what follows, we shall investigate superharmonic oscillation of order two and subsuperharmonic oscillation of order three-to-two of the system (2.6).

3 Superharmonic Solution $(2\Omega \cong \omega_o\ )$

In this case, we study subharmonic solution of order two-to-one, introduce the detuning parameters σ_1 to covert the small divisor term into secular terms

$$2\Omega=\omega_o+\epsilon\sigma_1, \tag{3.1}$$

and write

$$(\Omega-\omega_o)T_o=\omega_oT_o+\epsilon\sigma_1T_o=\omega_oT_o+\sigma_1T_1. \tag{3.2}$$

Inserting equations (3.2) into equation (2.6) and eliminating the terms that produce secular terms in u_1 yields the solvability condition

$$2\alpha A-2i\omega_oA'-2i\mu A\omega_o+12\alpha A^2\bar{A}-3A^2\alpha_2\bar{A}+(3A\bar{A}+\frac{1}{2})F_2e^{i\sigma_1T_1}=0. \tag{3.3}$$

Expressing A in the polar form

$$A(T_1)=\frac{1}{2}a(T_1)e^{i\beta(T_1)}. \tag{3.4}$$

Into Eq.(3.3) and separating the real and imaginary parts of equation (3.3), one obtains

$$\acute{a}=-a\mu+\frac{1}{\omega_o}(\frac{1}{2}+\frac{3a^2}{4})F_2\sin\psi. \tag{3.5}$$

$$a\psi'=a\sigma_1+\frac{a}{\omega_o}\alpha+(\frac{3\alpha}{2\omega_o}-\frac{3\alpha_2}{8\omega_o})a^3+\frac{1}{\omega_o}(\frac{1}{2}+\frac{3a^2}{4})F_2\cos\psi, \tag{3.6}$$

where

$$\psi=\sigma_1T_1-\beta. \tag{3.7}$$

It is obvious that, Eqs.(3.5) and (3.6) have a trivial solution which of corresponds to the trivial steady state solution. Nontrivial steady state solution correspond to the nontrivial fixed points (equilibrium points) of Eqs.(3.5) and (3.6). That is, they satisfy $\acute{a} = \acute{\psi} = 0$, and are given by

$$\frac{1}{2\omega_o}(1+\frac{3}{2}a^2)F_2\sin\psi = a\mu. \tag{3.8}$$

$$\frac{1}{2\omega_o}(1+\frac{3}{2}a^2)F_2\cos\psi = -(\sigma_1+\frac{\alpha}{\omega_o})a - (\frac{3\alpha}{2\omega_o}+\frac{3\alpha_2}{8\omega_o})a^3. \tag{3.9}$$

Equations (3.8) and (3.9)show that there are two possibilities: (trivial solution) at $a = 0$ and (nontrivial solution) at $a \neq 0$. Squaring and adding equations(3.8) and (3.9) we get the frequency-response equation

$$\sigma_1 = \frac{-8a^2\alpha\omega_o - 12a^4\alpha\omega_o + 3a^4\alpha_2\omega_o \pm 2\sqrt{4a^2F_2^2\omega_o^2 + 12a^4F_2^2\omega_o^2 + 9a^6F_2^2\omega_o^2 - 16a^4\mu^2\omega_o^4}}{8a^2\omega_o^2}. \tag{3.10}$$

Then, the first-order uniform expansion of the solution (first approximation) of Eq.(2.1) is given by

$$u = a\cos(2\Omega t - 2\psi) + O(\epsilon). \tag{3.11}$$

Stability analysis for the trivial solutions is equivalent to neglect the nonlinear terms solutions of equation (3.3) by neglecting the nonlinear terms we get

$$2\alpha A - 2i\omega_o A' - 2i\mu A\omega_o + \frac{1}{2}F_2 e^{i\sigma_1 T_1} = 0. \tag{3.12}$$

To determine the stability of the trivial steady state solution, it is convenient to rewrite A in the form

$$A = (B(T_1) + ib(T_1))e^{\frac{1}{2}i\sigma_1(T_1)}, \tag{3.13}$$

where B and b are real and imaginary parts and get

$$\acute{b} + \mu b + \Gamma_1 B = 0. \tag{3.14}$$

$$\acute{B} + \mu B - \Gamma_1 b = 0, \tag{3.15}$$

where $\Gamma_1 = \sigma_1 + \frac{\alpha}{\omega_o}$. Eqs.(3.14)and(3.15) admit solution of the form $(B, b) \propto (B, b)e^{\theta_o T_1}$, where (B, b) are constant. The eigenvalues of the coefficient matrix of Eqs.(3.14) and (3.15) are

$$\theta_o = -\mu \pm i\Gamma_1. \tag{3.16}$$

Then, the trivial solution is stable if the real parts of both eigenvalues are negative.

To determine the stability of the nontrivial steady state solutions given by Eqs.(3.8) and (3.9). Let

$$a = a_o + a_1(T_1) \qquad \& \qquad \psi = \psi_o + \psi_1(T_1). \tag{3.17}$$

Where a_o and ψ_o correspond to nontrivial steady state solutions and a_1 and ψ_1 are perturbations which are assumed to be small compared with a_o and ψ_o. Inserting equation (3.17) into equations (3.5) and (3.6) and linearizing the resulting equations, we obtain

$$\acute{a_1} = \mu a_1 - (\frac{a_o(8\alpha + 12a_o^2\alpha - 3a_o^2\alpha_2 + 8\sigma_1\omega_o)}{8\omega_0})\psi_1. \tag{3.18}$$

$$\begin{aligned}\acute{\psi_1} =& \frac{(16\alpha + 48a_o^2\alpha + 36a_o^4\alpha - 18a_o^2\alpha_2 - 9a_o^4\alpha_2 + 16\sigma_1\omega_o - 24a_o^2\sigma_1\omega_o)}{8(2a_o + 3a_o^3)\omega_o}a_1 \\ &- \frac{(16a_o\mu\omega_o - 24a_o^3\mu\omega_o)}{8(2a_o + 3a_o^3)\omega_o}\psi_1.\end{aligned} \tag{3.19}$$

Equations (3.18) and (3.19) admit solution of the form $(a_1, \psi_1) \propto (d_1, d_2)e^{\theta T_1}$ where (d_1, d_2) are constants. Provided that

$$\theta = -\frac{2\mu}{c_8} \pm \frac{1}{8}\sqrt{(\frac{1}{c_8^2\omega_o^2}(c_1\alpha^2 + c_2\alpha\alpha_2 + c_3\alpha_2^2 + c_4\alpha\sigma_1\omega_o + c_5\sigma_1\alpha_2\omega_o + c_6\mu^2\omega_0^2 + c_7\sigma_1^2\omega_o^2))}. \quad (3.20)$$

Where

$c_1 = -256 - 1536a^2 - 3456a^4 - 3456a^6 - 1296a^8, c_2 = 384a^2 + 1440a^4 + 1728a^6 + 648a^8,$
$c_3 = -108a^4 - 216a^6 - 81a^8, c_4 = -512 - 1536a^2 - 1152a^4, c_5 = 384a^2 + 576a^4,$
$c_6 = 576a^4, c_7 = -256 + 576a^4, c_8 = 2 + 3a^2.$
The solution is stable if and only if the real part of each of the eigenvalues of the coefficient of the matrix are less than or equal to zero.

4 Subsuperharmonic Solution $(3\Omega \cong 2\omega_o)$

In this section, we study subsuperharmonic solution of order three-to-one. To express the nearness of 3Ω to $2\omega_o$, one introduces the detuning parameter σ defined according to

$$3\Omega = 2\omega_o + \epsilon\sigma, \quad (4.1)$$

and writes

$$(3\Omega - 2\omega_o)T_o = 2\omega_o T_o + \epsilon\sigma T_o = 2\omega_o T_o + \sigma T_1. \quad (4.2)$$

Eliminating the secular terms form equation(3.2) yields

$$2\alpha A - 2i\omega_o A' - 2i\mu A\omega_o + 12\alpha A^2\bar{A} - 3A^2\alpha_2\bar{A} + \frac{3}{2}F_2\bar{A}^2e^{iT_1\sigma} = 0. \quad (4.3)$$

Using Eq.(3.4) into Eq.(4.3)and separating real and imaginary parts, we obtain the following modulation equations

$$\acute{a} = -a\mu + \frac{3}{8\omega_o}a^2F_2\sin\gamma. \quad (4.4)$$

$$\frac{1}{3}a\gamma' = \frac{1}{3}a\sigma - \frac{3a^3\alpha_2}{8\omega_o} + \frac{a\alpha}{\omega_o} + \frac{3a^3\alpha}{2\omega_o} + \frac{3}{8\omega_o}a^2F_2\cos\gamma, \quad (4.5)$$

where $\gamma = \sigma T_1 - 3\beta$. Substituting zero for $\acute{a}$ and $\acute{\gamma}$ into Eqs. (4.4)and(4.5) gives the following equations for the steady state solutions

$$\frac{3}{8\omega_o}a^2F_2\sin\gamma = a\mu. \quad (4.6)$$

$$\frac{3}{8\omega_o}a^2F_2\cos\gamma = -\frac{1}{3}a\sigma + \frac{3a^3\alpha_2}{8\omega_o} - \frac{a\alpha}{\omega_o} - \frac{3a^3\alpha}{2\omega_o}. \quad (4.7)$$

Eliminating the phase angle γ from equations (4.6) and (4.7) gives the expression for the solutions curves for the solution $a \neq 0$ as follows

$$(-\frac{1}{3}a_o\sigma + \frac{3a_o^3\alpha_2}{8\omega_o} - \frac{a_o\alpha}{\omega_o} - \frac{3a_o^3\alpha}{2\omega_o})^2 + (a_o\mu)^2 - (\frac{3}{8\omega_o}a_o^2F_2)^2 = 0, \quad (4.8)$$

i.e.

$$\sigma = \frac{3(-8\alpha\omega_o - 12a_o^2\alpha\omega_o + 3a_o^2\alpha_2\omega_o \pm \sqrt{9a_o^2F_2^2\omega_o^2 - 64\mu^2\omega_o^4})}{8\omega_o^2}. \quad (4.9)$$

Now, the stability analysis of the trivial solutions is determined as in the preceding section 3, so that we get the eigenvalues equation is similar to equation (3.16).

Following a procedure similar to that in section 3, one obtains the following eigenvalues that determine the stability of the

$$\theta = -\mu \pm \frac{\sqrt{\frac{576\alpha^2 - 1296a^4\alpha^2 + 648a^4\alpha\alpha_2 - 81a^4\alpha_2^2 + 384\alpha\sigma\omega_o + 768\mu^2\omega_o^2 + 64\sigma^2\omega_o^2}{\omega_o^2}}}{8\sqrt{3}}. \tag{4.10}$$

Consequently, a solution is stable if and only if the real parts of both eigenvalues (4.10) are less than or equal to zero.

5 Numerical Results

In this section the numerical solution of the frequency response equations (3.10) and (4.9) are studied. Frequency response equations (3.10) and (4.9) are nonlinear algebraic equations in the amplitude (a). The results are plotted in Figs. (1-15), which present the variation of amplitude (a) against the detuning parameter σ_1 and σ.

Figs. (1-8) represent the frequency response curves for superharmonic solution of order 2 for the parameters $[\omega_o = 2, \mu = 3, F_2 = 3, \alpha = 1, \alpha_2 = 2]$. In Fig. (1) for positive value of α, we note that the response amplitude has a stable single-valued curve and the maximum value exist at the point $\sigma_1 = -0.48$. For negative value of α, we observe that the maximum value shifts to the right so that the maximum value exist at the point $\sigma_1 = 0.57$, Fig. (2). When α takes the values 5 and 9, we note that the maximum shift to the left respectively so that the maximum values exist at the points $\sigma_1 = -2.79$ and $\sigma_1 = -5.06$, Fig. (3). For decreasing α with negative values (i.e. α take the value -5 and -9), we observe that the maximum shift to the right respectively so that the maximum values exist at the points $\sigma_1 = 2.79$ and $\sigma_1 = 5.06$, Fig. (4). When $\alpha_2 = 13$, we note that the singled-valued curves are intersect at the the same maximum value, Fig. (5). For increasing and decreasing the coefficient of nonlinear external excitation F_2 respectively, we observe that the singled-valued curves shift upward and downward respectively and have increasing and decreasing maximum values, Fig. (6,7,8).

Figs. (9-15) represent the frequency response curves for subsuperharmonic solution of order $\frac{3}{2}$ for the parameters $[\omega_o = 0.3, \mu = 0.2, F_2 = 3, \alpha = 0.01, \alpha_2 = 2]$. In Fig. (9) for positive values, we observe that the response amplitude has multivalued curve which consists of two branches while the lower branch has unstable solutions and the upper branch has stable solutions and there exist a saddle nodes bifurcations at the points $\sigma = -4.28$ and $\sigma = -4.34$. When α_2 takes the values 2 and 5, we observe that the multivalued curve contracted so that the upper and lower branches are shifts to downward so that these branches have decreased magnitudes respectively. The saddle nodes bifurcations exist at the points $\sigma = -4.30$ and $\sigma = -1.69$, Fig. (10). For decreasing α_2 with negative values (i.e. α_2 takes the values -2 and -5), we note that the multivalued curve is contracted so that the upper and lower branches have decreased magnitudes respectively and the saddle nodes bifurcations exist at the points $\sigma = 4.01$ and $\sigma = 1.46$, Fig. (11). As the parameter α is decreased with positive values (i.e. α takes the values 0.1 and 0.01), we get the same variation as in Fig. (10) so that the saddle nodes bifurcations exist at the points $\sigma = -6.23$ and $\sigma = -4.33$, Fig. (12). When the coefficient of nonlinear external excitation F_2 is decreased, we observe that the multivalued curve is contracted so that the upper and lower branches are shifts to downward and upward so that the upper branch has decreased magnitudes and the lower branch has increased magnitudes. As $F_2 = 0.3$ we observe that the multivalued curve is contracted and given semi-oval and the saddle nodes bifurcations exist at the points $\sigma = 2.01$ and $\sigma = 2.11$, Fig. (13). For increasing the damping factor μ, we note that the multivalued curve is contracted and the saddle nodes bifurcations exist at the points $\sigma = -3.79$ and $\sigma = 3.89$, Fig. (14). When the natural frequency ω_o takes the values 0.9 and 2, we observe that the multivalued curve is contracted respectively so that the upper branch has stable and unstable solutions while the lower branch has stable and unstable solutions and these branches are intersect at the point $\sigma = -4.31$. The saddle nodes bifurcations exist at the points $\sigma = -1.40$ and $\sigma = -0.49$, Fig. (15).

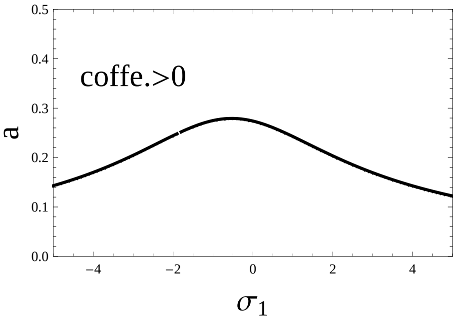

Fig. 1

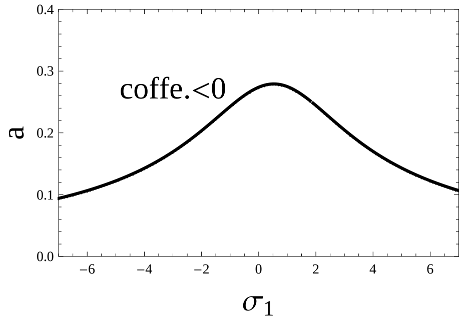

Fig. 2

Figs. 1 and 2. The frequency response curves of the superharmonic solution of order 2 for the parameters $\omega_o = 2, \mu = 3, F_2 = \pm 3, \alpha = \pm 1, \alpha_2 = \pm 2$.

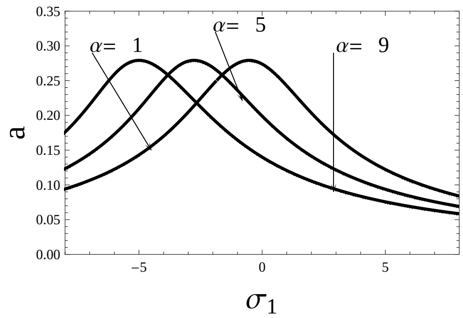

Fig. 3

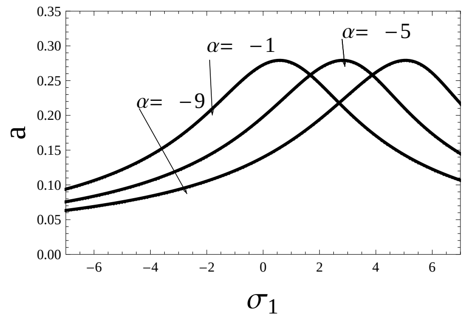

Fig. 4

Figs. 3 and 4. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing α.

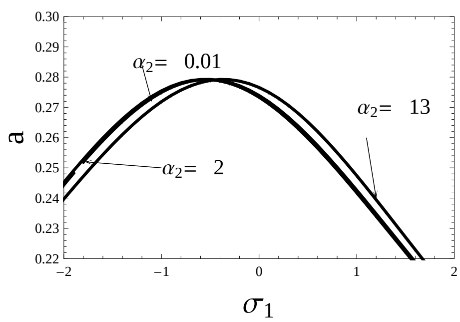

Fig. 5

Fig. 5. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing α_2.

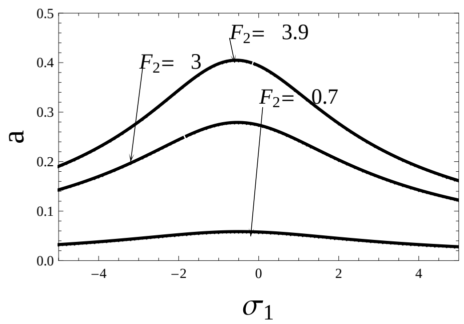

Fig. 6

Fig. 6. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing F_2.

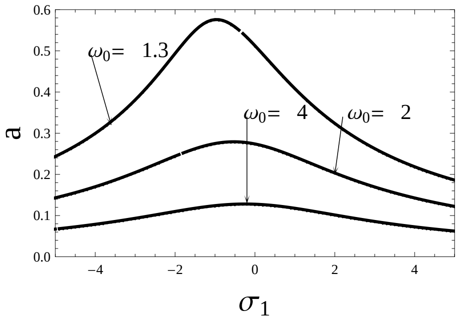

Fig. 7

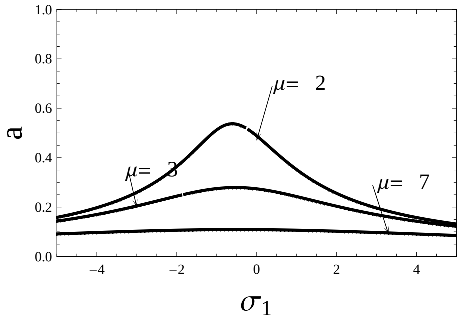

Fig. 8

Fig. 7. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing ω_o.

Fig. 8. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing μ.

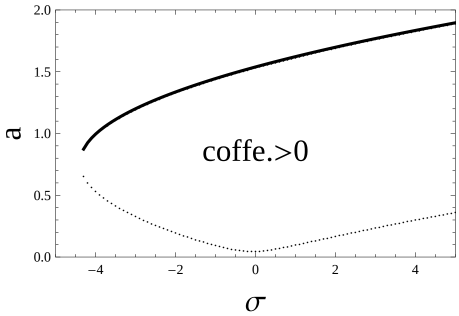

Fig. 9

Fig. 9. The frequency response curves of the subsuperharmonic solution of order $\frac{3}{2}$ for the parameters $\omega_o = .3, \mu = 0.2, F_2 = 3, \alpha = .01, \alpha_2 = 2$.

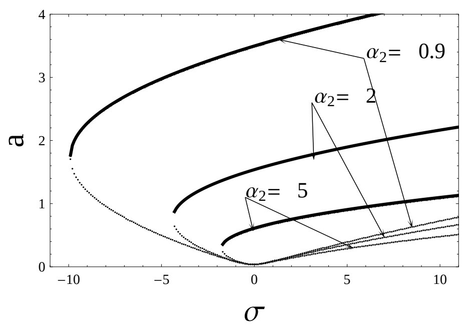

Fig. 10

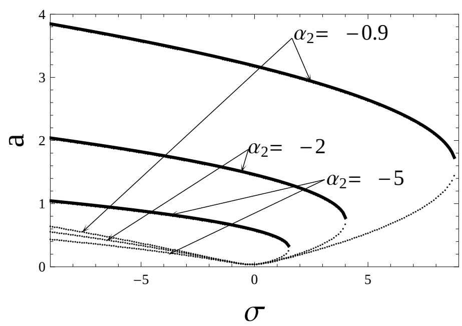

Fig. 11

Figs. 10 and 11. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing α_2.

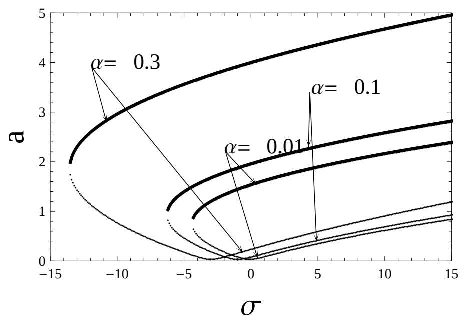

Fig. 12

Fig. 12 Variation of the amplitude of the response with the detuning parameter for increasing and decreasing α.

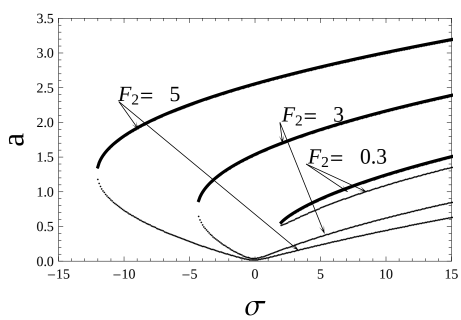

Fig. 13

Fig. 13. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing F_2.

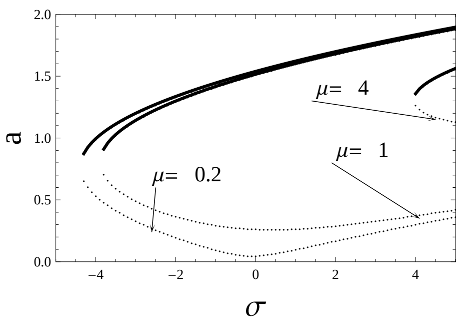

Fig. 14

Fig. 14. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing μ.

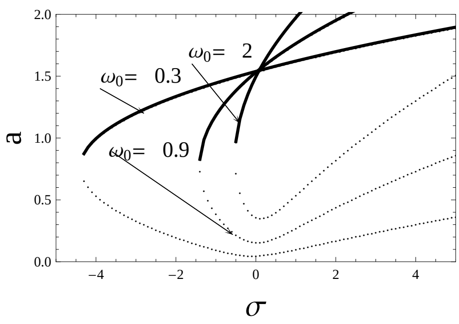

Fig. 15

Fig. 15. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing ω_o.

6 Summary and Conclusion

An analytical and numerical technique is used to predict the qualitative change taking place in the stable solutions of the non-linear modified Duffing equation subjected to a bi-harmonic parametric and external excitations. The multiple time scales are used to investigate a first-order approximate analytical solution. The modulation equations (reduced equations) of the amplitude and phase are obtained. Steady state solutions and their stability condition are determined. The following conclusions can be deduced from the analysis:

From the frequency-response curves of superharmonic oscillation of order two(2), we note that the response amplitude has a single-valued curve and all solutions are stable. The maximum value shifts to the left and right for increasing and decreasing with decreasing α with negative values respectively. The maximum value shifts upward for increasing F_2, ω_o and for decreasing μ. The maximum value shifts downward for decreasing F_2 and for increasing ω_o and μ.

From the frequency-response curves of subsuperharmonic oscillation of order $\frac{3}{2}$, we observe that the response amplitude has multivalued curve. The stable and unstable solutions are exist in the upper and lower branches respectively. For positive (negative) values, we note that the multivalued curve bents to the right (left) and hard (soft) nonlinearities. When $F_2 = 0.3$ and $\mu = 4$ we observe that the multivalued curve contracted and given semi-ovals. The upper branch of the multivalued curves are intersect at the same point $\sigma = -4.31$, when ω_o takes the values 0.3, 0.9 and 2.

Competing Interests

Authors have declared that no competing interests exist.

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